2. HOMOTOPY

2.1 Basic Definitions

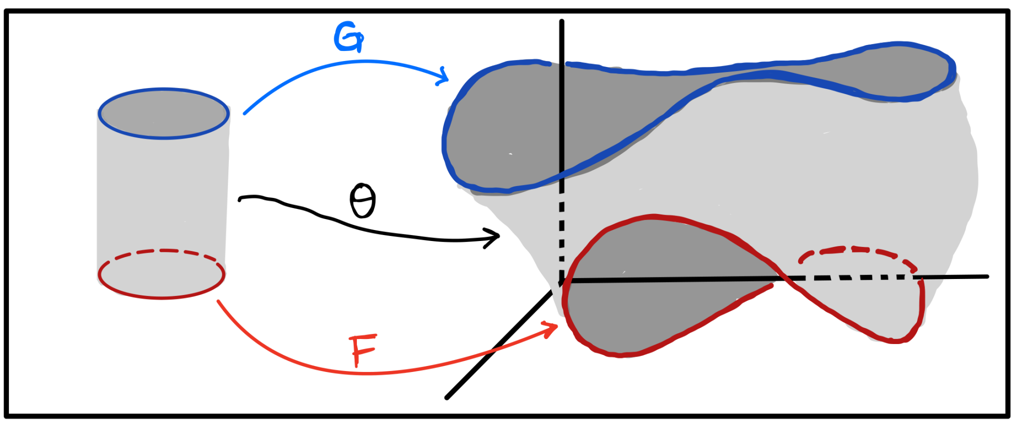
Given topological spaces X and Y, the set of all continuous functions from X to Y is typically quite large and complicated even in relatively simple cases (e.g., when both X and Y are the unit circle in $\mathbb{R}^2$). In order to study such functions, we are compelled to define interesting equivalence relations on them and restrict attention to equivalence classes. Among the deepest and most fruitful equivalence relations between functions $X \to Y$ is the notion of a *homotopy*.

Definition 2.1. Two continuous functions $F, G : X \to Y$ between topological spaces X and Y are **homotopic** if there is a third continuous function

$$\theta : X \times [0,1] \to Y$$

(called a **homotopy**) so that for all x in X, we have $\theta(x,0) = F(x)$ and $\theta(x,1) = G(x)$.

The requirement that θ also be continuous is absolutely essential here, since it is always possible to find discontinuous θ satisfying the requirements of this definition. Thus, the fundamental idea behind this definition is to put two functions in the same equivalence class whenever you can continuously interpolate from one to the other as a parameter $t \in [0,1]$ slides from 0 to 1. The picture below illustrates the homotopy equivalence of two maps F, G when X is a circle and Y is $\mathbb{R}^3$. These are homotopic if we can find a continuous θ from the cylinder $X \times [0,1]$ to $\mathbb{R}^3$ whose restriction to the lower boundary $X \times \{0\}$ coincides with F and restriction to the upper boundary $X \times \{1\}$ coincides with G.

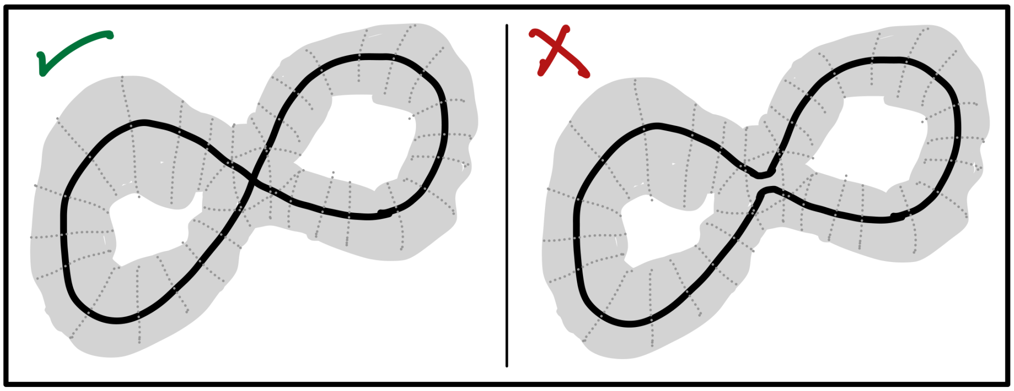


Homotopies between functions can be used in order to produce an equivalence relation on topological spaces as well.

Definition 2.2. Two topological spaces X and Y are **homotopy equivalent** if there are continuous maps $F : X \to Y$ and $G : Y \to X$ so that

(1) the composite $F \circ G$ is homotopic to the identity map on Y, while
(2) the composite $G \circ F$ is homotopic to the identity map on X.

A pair of continuous maps F and G satisfying the two conditions above are often called *homotopy inverses* of each other, although it is important to note that in general there is no uniqueness of such inverses — the set of homotopy inverses for a given F might contain several maps. Homotopy equivalence is a topological property that tends to be largely agnostic to metric information. The two panels below are designed to illustrate this phenomenon: in the first case, the 2-dimensional thickened figure-8 is homotopy equivalent to the thinner 1-dimensional figure-8 in its interior. But if we perturb this thinner curve ever so slightly to create a single loop, then homotopy equivalence no longer holds.



Two simplicial complexes K and L are said to be homotopy equivalent, or have the same *homotopy type*, whenever their geometric realizations $|K|$ and $|L|$ are homotopy equivalent in the sense of the definition above. It may not be immediately obvious that homotopy is an important equivalence relation between topological spaces — absorbing this fact takes time and experience. What should be clearer even at this early stage is that homotopy equivalence is far less rigid than homeomorphism: homeomorphic spaces are always homotopy equivalent, but the converse does not hold.

2.2 Contractible Spaces

The quest to study topological spaces up to homotopy equivalence has a natural starting point — we begin by asking which spaces are the least complicated from a homotopical perspective.

DEFINITION 2.3. A topological space X is **contractible** if it is homotopy equivalent to the one-point space.

You should check that X is contractible if and only if there exists some point $p \in X$ so that the identity map on X is homotopic to the constant map sending every point of X to p. In particular, the empty set $\varnothing$ is *not* contractible.

EXAMPLE 2.4. Here are several families of contractible simplicial complexes:

(1) **Solids**: for each $k \geq 0$ the solid k-simplex $\Delta(k)$ is contractible.
(2) **Cones**: the cone over *any* simplicial complex K (see Definition 1.19) is contractible.
(3) **Trees**: a tree is a connected graph with no cycles; these are all contractible.

We will prove the contractibility of these after developing some helpful machinery. For now, it is important to start building a mental database which contains as many contractible spaces as possible. The next few sections contain a suite of extremely powerful tools for detecting homotopy equivalence, and all of these tools rely in one way or another on your ability to recognize contractible spaces. The underlying reason for this dependence is the following vital result.

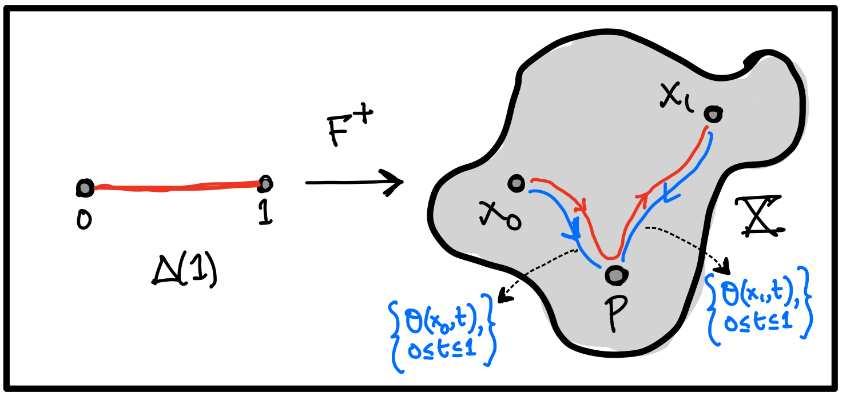
LEMMA 2.5. *Let X be a topological space and $k \geq 1$ an integer. If X is contractible, then any continuous map $F : |\partial\Delta(k)| \to X$ from the hollow k-simplex to X can be extended to a continuous map $F^+ : |\Delta(k)| \to X$ from the solid k-simplex.*

PROOF. Even the case $k = 1$ is quite insightful, so we will go over it carefully. Since $|\Delta(1)|$ is homeomorphic to the unit interval $[0,1]$ and $|\partial\Delta(1)|$ consists of the endoints $\{0,1\}$, we must show that X is *path-connected*, i.e., given any pair of points $F(0) = x_0$ and $F(1) = x_1$ in X, there is a continuous path in X from x_0 to x_1.

From Definition 2.3, we know that X is contractible if and only if there is some point $p \in X$ so that the identity map on X is homotopic to the constant map $X \to p$. Thus, there exists a homotopy $\theta : X \times [0,1] \to X$ satisfying $\theta(x,0) = x$ and $\theta(x,1) = p$ for all x in X. As we vary t from 0 to 1 for any given x in X, we obtain a continuous path $\theta(x,t)$ from x to our special point p — in particular both x_0 and x_1 admit paths to p. Thus, we can concatenate these two paths to get a path from x_0 to x_1 that passes through p; more explicitly, the desired extension $F^+ : [0,1] \to X$ is given in terms of θ by the piecewise-formula

$$F^+(t) = \begin{cases} \theta(x_0, 2t) & t \leq 1/2 \\ \theta(x_1, 2t-1) & t > 1/2. \end{cases}$$

This extension is continuous because at $t = 1/2$ both pieces are guaranteed to equal p. The following picture may help if the numerology of this formula is mysterious.



The argument for $k \geq 2$ is more technical and subscript-infested, but the basic principle remains the same — homotopies to constant maps allow us to "fill in" the F-images of hollow simplices to produce F^+-images of the corresponding solid simplices. □

In the argument above, we used a homotopy θ to define an extension map without ever having an explicit formula for θ; this is quite typical because in general homotopies can get quite complicated even when relating simple maps between benign spaces. One refreshing exception to this unfortunate state of affairs is provided by the class of *straight-line homotopies*: given maps $f, g : X \to Y$ with $Y \subset \mathbb{R}^n$, one often attempts to use $\theta(x,t) = t \cdot f(x) + (1-t) \cdot g(x)$. Of course, there is no guarantee that the image of such a θ will actually lie in Y. Our next result highlights an important instance where this straight-line strategy succeeds.

PROPOSITION 2.6. *For each dimension $k \geq 0$, the solid k-dimensional simplex $\Delta(k)$ is contractible.*

PROOF. Let $\{x_0, \ldots, x_k\} \subset \mathbb{R}^n$ be any set of affinely independent points, so the geometric realization of $\Delta(k)$ is given (up to homeomorphism) by

$$|\Delta(k)| = \left\{ \sum_{i=0}^{k} t_i x_i \mid t_i \geq 0 \text{ and } \sum_{i=0}^{k} t_i = 1 \right\}.$$

Now consider the continuous map $\theta : |\Delta(k)| \times [0,1] \to \Delta(K)$ that sends each $x = \sum_{i=0}^{k} t_i x_i$ in $|\Delta(k)|$ and t in $[0,1]$ to the point

$$\theta(x,t) = [1 - t(1-t_0)] \cdot x_0 + t \cdot \sum_{i=1}^{k} t_i x_i.$$

This formula prescribes a straight-line homotopy between the identity map (at $t = 1$) and the constant map (at $t = 0$) sending everything to x_0. Three routine verifications have been left as exercises: to complete the proof, one must show that $\theta(x, t)$ lies in $|\Delta(k)|$ for all t, that $\theta(x, 0)$ is just the constant map to x_0, and that $\theta(x, 1)$ is the identity map on $|\Delta(k)|$. $\square$

Armed with knowledge of many contractible spaces, we are ready to explore a suite of homotopy equivalence detectors.

2.3 Carriers

Let K be a simplicial complex and X a topological space.

DEFINITION 2.7. A **carrier** C for K in X is an assignment of a subset $C(\sigma) \subset X$ to every simplex σ of K so that $C(\sigma) \subset C(\tau)$ holds whenever σ is a face of τ.

We say that C **carries** a continuous map $F : |K| \to X$ if for each simplex $\sigma \in K$ we have $F(|\sigma|) \subset C(\sigma)$. Similarly, we say that C carries a homotopy $\theta : |K| \times [0, 1] \to Y$ if for each intermediate t in $[0, 1]$ the map $\theta_t : |K| \to X$ given by

$$\theta_t(x) = \theta(x, t)$$

is carried by C in the sense described above. The next result is among the most powerful and widely-applicable tools for testing whether two maps $|K| \to X$ are homotopic.

LEMMA 2.8. **(The Carrier Lemma)** *Let C be a carrier for K in X. If the subset $C(\sigma) \subset X$ is contractible for each simplex $\sigma \in K$, then* **(a)** *there exists a continuous map $F : |K| \to X$ carried by C;* **(b)** *any two continuous maps $F, G : |K| \to X$ carried by C are homotopic; and* **(c)** *in fact, we can always choose a homotopy $\theta : |K| \times [0, 1] \to X$ between F and G that is also carried by C.*

PROOF. Index the simplices of K as $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ so that the faces of each simplex have lower indices than that simplex itself — this can be ensured for instance by indexing all the 0-dimensional simplices before all the 1-dimensional simplices, and so forth. There is now a filtration $\{\mathbf{S}_i K \mid 1 \leq i \leq m\}$ of K (see Definition 1.6) obtained by setting

$$\mathbf{S}_i K = \bigcup_{j \leq i} \sigma_j.$$

We will show **(b)** and **(c)** by induction on i; the argument for **(a)** is eerily similar and has been assigned as an exercise.

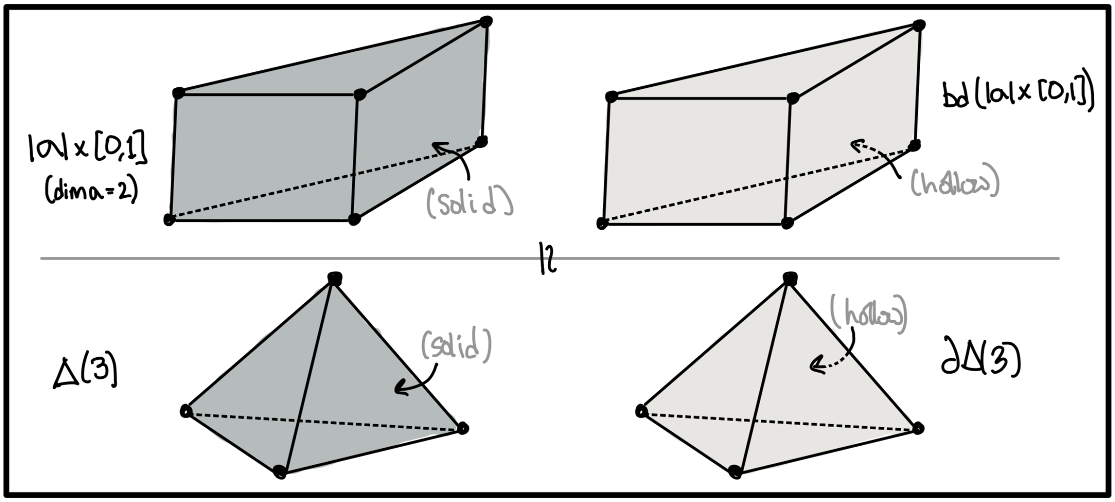
Base case: When $i = 1$ we must have a simplex σ_1 of minimum dimension, i.e., a vertex. By the hypotheses of this Theorem, the maps F and G send our vertex σ_1 to possibly distinct points (let's call them x_0 and x_1) in the contractible set $C(\sigma_1) \subset X$. The points x_0 and x_1 are evidently the image of a map $|\partial\Delta(1)| \to C(\sigma)$, so by Lemma 2.5 there is a path lying in $C(\sigma_1)$ from x_0 to x_1. This path prescribes a homotopy carried by C between the restrictions of F and G to $S_1 K = \sigma_1$.

Inductive step, part 1: Now let us assume that for some $i > 1$ the restrictions of F and G to $S_{i-1} K \subset K$ admit a homotopy $\theta : |\mathbf{S}_{i-1} K| \times [0, 1] \to X$ carried by C. We must extend this θ continuously to the larger space $|\mathbf{S}_i K| \times [0, 1]$; thus it suffices to define θ on the subset $|\sigma_i| \times [0, 1]$, where σ_i is the unique simplex satisfying $S_i = S_{i-1} \cup \sigma_i$. Let $B \subset |\mathbf{S}_{i-1}|$ be the union of geometric realizations of all the faces $\tau \leq \sigma_i$ other than σ_i itself. Since all the $C(\tau)$ are subsets of $C(\sigma_i)$ by Definition 2.7, we note that the image $\theta(B \times [0, 1])$ is entirely contained within $C(\sigma_i)$. Moreover, by the requirement that C carries F and G, both $F(|\sigma_i|)$ and $G(|\sigma_i|)$ also lie inside $C(\sigma_i)$.

Inductive step, part 2: The key observation here is as follows: writing $d = \dim \sigma_i$, the product $|\sigma_i| \times [0,1]$ is homeomorphic to $|\Delta(d)| \times [0,1]$, which in turn is homeomorphic to $|\Delta(d+1)|$. Consequently, the boundary[1] of $|\sigma_i| \times [0,1]$ is homeomorphic to the subset

$$|\partial\Delta(d+1)| \simeq \Big(|\partial\Delta(d)| \times [0,1]\Big) \cup \Big(|\Delta(d)| \times \{0,1\}\Big).$$

Here is a figure illustrating these spaces for $d = 2$:



Now the first piece of this union $|\partial\Delta(d)| \times [0,1]$ is homeomorphic to $B \times [0,1]$ while the second piece is homeomorphic to two disjoint copies of $|\sigma_i|$. Our homotopy θ sends the first piece to $C(\sigma_i)$ by **part 1** of the inductive step. As for the second piece, we know that

$$\theta(|\sigma_i|, 0) = F(|\sigma_i|) \subset C(\sigma_i).$$

Here the equality follows from Definition 2.1 while the containment is a consequence of the assumption that C carries F. Similarly, we also have $\theta(|\sigma_i|, 1) = G(|\sigma_i)| \subset C(\sigma_i)$. So up to homeomorphism, θ constitues a map from the entire boundary $\partial\Delta(d+1)$ to the contractible set $C(\sigma_i) \subset X$. Lemma 2.5 guarantees a continuous extension $\theta^+ : |\Delta(d+1)| \to C(\sigma_i)$, and using the homeomorphism $\Delta(d+1) \simeq |\sigma_i| \times [0,1]$ gives us the desired continuous extension of θ to $|\sigma_i| \times [0,1]$. ☐

The utility of the Carrier lemma in homotopically-oriented problems is difficult to overstate. Here is a simple consequence designed to work directly with simplicial maps. We say that two simplicial maps $f, g : K \to L$ are **contiguous** if for any simplex σ of K, the union $f(\sigma) \cup g(\sigma)$ is a simplex of L.

COROLLARY 2.9. *If $f, g : K \to L$ are contiguous, then they must be homotopic.*

PROOF. For each simplex σ in K, let $C(\sigma) \subset |L|$ be the geometric realization of the union-simplex $f(\sigma) \cup g(\sigma)$. This assignment C prescribes a carrier for K in $|L|$; clearly, C carries both f and g. And finally, since solid simplices are contractible by Proposition 2.6, the desired conclusion follows from Lemma 2.8 **(b)**. ☐

This result has satisfying and immediate applications: for instance, we can now easily show that $\text{Cone}(K)$ is contractible for any simplicial complex K. Writing v_* for the additional vertex as in Definition 1.19, apply Corollary 2.9 to the case where f is the identity map on $\text{Cone}(K)$ while g is the map sending every vertex to v_*.

[1]Here we have used the fact that the boundary of a product $\text{bd}(P \times Q)$ is the union $(P \times \text{bd}\, Q) \cup (\text{bd}\, P \times Q)$.

2.4 FIBERS

Let $f : K \to L$ be a simplicial map; for each simplex τ in L, the **fiber of f under τ** is the collection of simplices in K given by

$$\tau/f = \{\sigma \in K \mid f(\sigma) \leq \tau\}. \tag{2}$$

Each such fiber is a subcomplex of K; and moreover, τ/f is a subcomplex of τ'/f whenever $\tau \leq \tau'$ in L. We will use the Carrier lemma three times below to show that simplicial maps with contractible fibers induce homotopy equivalences — this forms a special case of a far more general result called *Quillen's Theorem A*.

THEOREM 2.10. **(Quillen's Fiber Theorem)** *Let $f : K \to L$ be a simplicial map. If the fiber τ/f is contractible for every simplex τ in L, then the induced continuous map $|f| : |K| \to |L|$ admits a homotopy inverse $G : |L| \to |K|$; and in particular, K and L are homotopy equivalent.*

PROOF. For each simplex τ of L, let $C(\tau) \subset |K|$ be the geometric realization of the fiber τ/F; this provides a carrier for L in $|K|$ with each $C(\tau)$ contractible, so by Lemma 2.8 **(a)** we know that there exists a continuous $G : |L| \to |K|$ satisfying $G(|\tau|) \subset C(\tau) = |\tau/f|$ for all τ in L. We will confirm that any such G is a homotopy inverse for $|f|$.

1. $|f| \circ G$ **is homotopic to the identity on** L**:** for each simplex τ in L, we have the containment

$$|f| \circ G(|\tau|) \subset |\tau|,$$

simply because $G(|\tau|)$ is contained in $|\tau/f|$. Therefore, the assignment $C_L(\tau) = |\tau|$ prescribes a carrier (for L in $|L|$) which carries both $|f| \circ G$ and the identity map on L. Since each $|\tau|$ is contractible by Proposition 2.6, we have from Lemma 2.8 **(b)** that $|f| \circ G$ is homotopic to the identity on L as desired.

2. $G \circ |f|$ **is homotopic to the identity on** K**:** for each simplex σ in K, we know from Proposition 1.11 that the $|f|$-image of $|\sigma|$ is exactly $|f(\sigma)| \subset |L|$. Recall that by our construction of G, we have the containment

$$G(|f(\sigma)|) \subset C(f(\sigma)) = |f(\sigma)/f|.$$

So if we define C_K to be the carrier for K in $|K|$ given by $C_K(\sigma) = |f(\sigma)/f|$, we know that C_K carries $G \circ |f|$. Note also that σ automatically lies in $f(\sigma)/f$ by (2), so C_K also carries the identity map on K. Since each $C_K(\sigma)$ is contractible by our assumption on the fibers of f, a final appeal to Lemma 2.8 **(b)** concludes the argument. ☐

The strength of Quillen's fiber theorem lies in the fact that it allows us to conclude homotopy equivalence of simplicial complexes K and L given only a one-way simplicial map $f : K \to L$. As long as this f has contractible fibers, one is not required to painstakingly construct an explicit homotopy inverse $|L| \to |K|$.

2.5 NERVES

A finite **open cover** $U_\bullet$ of a topological space X is a collection of open subsets $U_\alpha \subset X$ (here α ranges over some finite index set A) satisfying

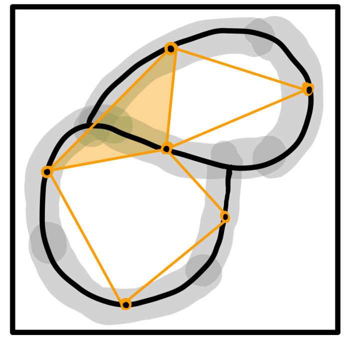
$$X = \bigcup_{\alpha \in A} U_\alpha.$$

By keeping track of how the different U_α intersect one another, we can build a simplicial complex on the vertex set A; the hope is to appropriately constrain the cover so that this simplicial complex is homotopy equivalent to X.

DEFINITION 2.11. The **nerve** $\mathbf{N}(\mathcal{U}_\bullet)$ of an open cover $\{\mathcal{U}_\alpha \mid \alpha \in A\}$ of a topological space X is the simplicial complex whose i-simplices are given by all subsets $\sigma \subset A$ of cardinality $(i+1)$ for which the intersection

$$\mathbf{Supp}(\sigma) := \bigcap_{\alpha \in \sigma} \mathcal{U}_\alpha$$

is nonempty.



This intersection $\mathbf{Supp}(\sigma) \subset X$ is called the *support* of the simplex σ, and those encountering this notion for the first time should beware that $\sigma \leq \tau$ in $\mathbf{N}(\mathcal{U}_\bullet)$ means $\mathbf{Supp}(\sigma) \supset \mathbf{Supp}(\tau)$ as subsets of X. In particular, the vertices of $|N(\mathcal{U}_\bullet)|$ have the larger supports than the edges which admit them as faces, and so on.

Having gone through the effort of finding an open cover $\mathcal{U}_\bullet$ of a topological space X, one wonders to what extent the homotopy type of X is captured by the geometric realization $|N(\mathcal{U}_\bullet)|$ of the associated nerve. The task appears absolutely hopeless at first glance — for instance, we could always choose $\mathcal{U}_\bullet$ to consist of a single subset $\mathcal{U}_1 = X$, in which case its nerve is just $\Delta(0)$ regardless of X. As with most of the other results described here, the key to solving this problem is contractibility. If we require all nonempty supports to be contractible subsets of X, then the following miracle occurs.

THEOREM 2.12. **(The Nerve theorem)** *Let $\{\mathcal{U}_\alpha \mid \alpha \in A\}$ be a finite open cover of a topological space X. If each simplex $\sigma \in N(\mathcal{U}_\bullet)$ has contractible support $\mathbf{Supp}(\sigma) \subset X$, then $|N(\mathcal{U}_\bullet)|$ is homotopy equivalent to X.*

PROOF. Let $X(\mathcal{U}_\bullet)$ be the subset of the product $X \times |N(\mathcal{U}_\bullet)|$ containing all pairs (x, u) for which there is a simplex σ in $N(\mathcal{U}_\bullet)$ satisfying both $x \in \mathbf{Supp}(\sigma)$ and $u \in |\sigma|$. There are natural projection maps from $X(\mathcal{U}_\bullet)$ to both X and $|N(\mathcal{U}_\bullet)|$:

$$\begin{tikzcd} & X(\mathcal{U}_\bullet) \arrow[dl, two heads, "p"'] \arrow[dr, two heads, "q"] & \\ X & & |N(\mathcal{U}_\bullet)| \end{tikzcd}$$

In particular, $p(x, u) = x$ and $q(x, u) = u$ for every (x, u) in $X(\mathcal{U}_\bullet)$. Next, we show that for each point x in X and u in $|N(\mathcal{U}_\bullet)|$, the fibers $p^{-1}(x)$ and $q^{-1}(u)$ are contractible subsets of $X(\mathcal{U}_\bullet)$.

1. Fibers of p: For each point x in X, the fiber $p^{-1}(x) \subset X(\mathcal{U}_\bullet)$ is homeomorphic to the set of all $u \in |N(\mathcal{U}_\bullet)|$ lying in the realizations of simplices σ whose supports contain x. But all such σ must be faces of the single simplex σ_x in $N(\mathcal{U}_\bullet)$ whose support is the intersection of *all* $\mathcal{U}_\alpha$ satisfying $x \in \mathcal{U}_\alpha$. Thus, $p^{-1}(x)$ is homeomorphic to the geometric realization of $|\sigma_x|$, which must be contractible by Proposition 2.6.

2. Fibers of q: Given u in $|N(\mathcal{U}_\bullet)|$, let $\sigma_u \in N(\mathcal{U}_\bullet)$ be the unique simplex containing u in the interior of its realization. The fiber $q^{-1}(u) \subset X(\mathcal{U}_\bullet)$ is homeomorphic to the support $\mathbf{Supp}(\sigma_u) \subset X$, which is contractible by assumption.

3. Finale: There is a variant of Theorem 2.10 which applies to a large class of continuous (not necessarily simplicial) maps between metric spaces (not necessarily simplicial complexes). In particular, this result implies that sufficiently well-behaved maps — such as our p and q — induce homotopy equivalences if their fibers over all *points* of their codomains are contractible[2].

[2]For details, see the main result of S Smale's 1957 paper *A Vietoris Mapping Theorem for Homotopy*.

An appeal to this modified fiber theorem establishes that X and $|N(\mathcal{U}_\bullet)|$ are both homotopy equivalent to $X(\mathcal{U}_\bullet)$ via p and q respectively, so the desired conclusion follows. $\square$

There are at least three things to be noted about the Nerve theorem and its proof. First, it was really convenient to have a fiber theorem at our disposal — not only did we avoid having to build any homotopic inverses, but we even managed to avoid building a one-way map relating X to $|N(\mathcal{U}_\bullet)|$. Second, the Nerve theorem gives us a mechanism for going back from topological spaces to simplicial complexes; in that sense, it constitutes a sort of converse to geometric realizations from Definition 1.7. And third, this theorem guarantees that Čech filtrations from Definition 1.16 accurately capture the homotopy type of the underlying union of balls at each scale.

COROLLARY 2.13. *Let $M \subset \mathbb{R}^n$ be a finite set of points. For each radius $\epsilon > 0$, the union $M^{+\epsilon} \subset \mathbb{R}^n$ of radius ϵ Euclidean balls around the points of M is homotopy equivalent to the geometric realization of the Čech complex $\mathbf{C}_\epsilon(M)$.*

PROOF. For each point x in M, let $B_\epsilon(x)$ be the open ball of radius ϵ around x. By definition of $M^{+\epsilon}$, we have

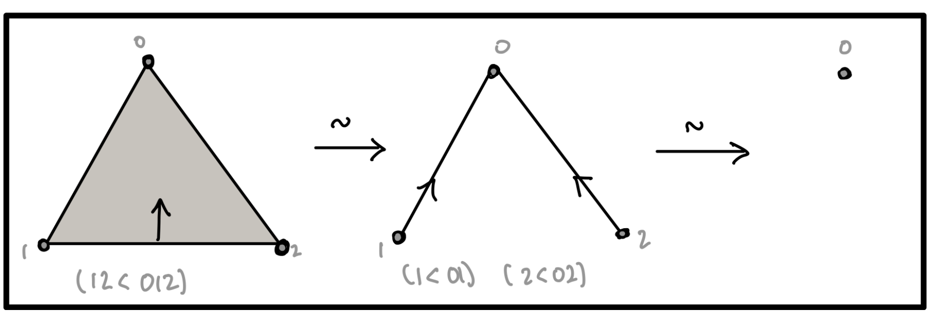
$$M^{+\epsilon} = \bigcup_{x \in M} B_\epsilon(x),$$

so the collection $\{B_\epsilon(x) \mid x \in M\}$ constitutes an open cover of $M^{+\epsilon}$. The Čech complex $\mathbf{C}_\epsilon(M)$ is precisely the nerve of this cover, so the desired conclusion follows from the Nerve theorem if we can show that nonempty intersections of Euclidean balls are contactible. Such intersections are always *convex* subsets of $\mathbb{R}^n$, and their contractibility will be established in one of the Exercises of this Chapter. $\square$

There are no homotopical guarantees analogous to the above result which apply to the Vietoris-Rips filtration.

2.6 ELEMENTARY COLLAPSES

There is a simple combinatorial operation on simplicial complexes which allows us to find homotopy-equivalent subcomplexes by performing a series of moves; each such move removes two adjacent simplices $(\sigma < \tau)$ at a time, and has a very concrete and algorithmic flavor. For instance, one can show that $\Delta(2)$ is contractible simply by drawing the following diagram:

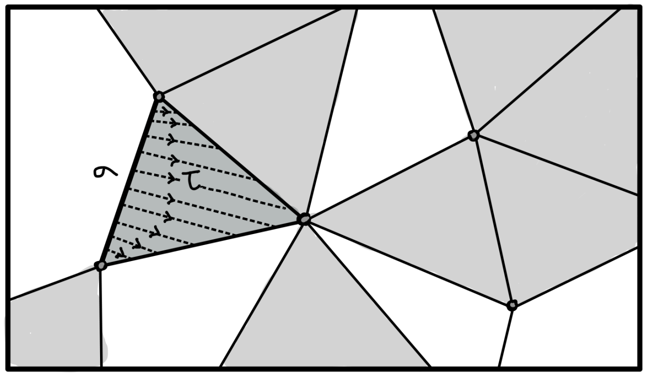


Our goal in this section is to describe these homotopy-preserving moves.

Let K be a simplicial complex. We call two distinct simplices $(\sigma < \tau)$ of K a **free face pair** if the open star of σ (see Definition 1.17) satifies $\mathbf{st}_K(\sigma) = \{\sigma, \tau\}$. For such a pair we immediately have $\dim \tau = \dim \sigma + 1$; moreover, there can be no other simplices in K (besides σ and τ) which admit σ as a face.

PROPOSITION 2.14. *If $(\sigma < \tau)$ is a free face pair in K, then the collection $K' = K - \{\sigma, \tau\}$ forms a subcomplex of K, and in fact K' is homotopy equivalent to K.*

PROOF. Assume for the sake of contradiction that some simplex γ in K' is missing a face; such a γ would have to satisfy $\gamma > \sigma$ in K; this forces $\mathbf{st}_K(\sigma)$ to contain γ and violates our free face assumption. Thus, $K' \subset K$ is a subcomplex. To see the desired homotopy equivalence to K, consider the following figure:



There is a map $r : |K| \to |K'|$ which is the identity away from $|\tau|$ and sends all points of $|\tau|$ along straight line segments to points in the union $\bigcup_{\sigma \neq \eta < \tau} |\eta|$ of realizations of all faces of τ except σ. This map serves as a homotopy inverse to the inclusion $i : K' \hookrightarrow K$; on the one hand, the composite $r \circ |i|$ is the identity map on $|K'|$. And on the other hand, these straight line segments generate a homotopy from $|i| \circ r$ to the identity map on K. □

The removal of a free face pair $(\sigma < \tau)$ from K is called an **elementary collapse**. These can be iterated, as shown in our diagrammatic reduction of $\Delta(2)$ to $\Delta(0)$ drawn above. One important point to note, visible already in the figure above, is that the subcomplex $K' = K - \{\sigma, \tau\}$ might contain free face pairs that were unavailable in K: when we remove the pair $(12 < 012)$ from $\Delta(2)$, the pairs $(1 < 01)$ and $(2 < 02)$ become free and can be safely removed in the second step. We say that K **collapses** onto a subcomplex L if there is a filtration (as in Definition 1.6) of the form

$$L = \boldsymbol{F}_1 K \subset \boldsymbol{F}_2 K \subset \cdots \subset \boldsymbol{F}_n K = K$$

where each $\boldsymbol{F}_i K$ is obtained by removing a single free-face pair from the subsequent $\boldsymbol{F}_{i+1} K$. By Proposition 2.14, all the $\boldsymbol{F}_i K$ are homotopy equivalent to each other in this case. Thanks to their simple combinatorial nature, elementary collapses can be algorithmically implemented on a computer.

2.7 BONUS: SIMPLICIAL APPROXIMATION

The contents of the section are not used elsewhere in this text; they have been included here because Theorem 2.15 described below is a foundational result in simplicial algebraic topology. It allows us to study homotopy classes functions between (geometric realizations of) simplicial complexes using only simplicial maps rather than arbitrary continuous ones.

Here is a fairly natural challenge in light of our quest to understand simplicial complexes up to homotopy equivalence.

> *Assume that $F : |K| \to |L|$ is a continuous map between the geometric realizations of two simplicial complexes K and L. Does there exist a simplicial map $f : K \to L$ so that $|f|$ is homotopic to F?*

Unfortunately, the answer to this question as stated is *no*. One way to see why (without doing any heavy computations) is to note that the set of all simplicial maps $K \to L$ is always finite, so it is unreasonable to expect simplicial maps to attain all possible homotopy types achievable by the (typically *very* infinite) set of continuous maps $|K| \to |L|$. The good news, however, is that the answer to our challenge becomes yes *if* we give ourselves the ability to barycentrically subdivide the domain K finitely many times (as described in Definition 1.12). The following result is called the **simplicial approximation theorem**.

THEOREM 2.15. *Let $F : |K| \to |L|$ be a continuous map between the geometric realizations of two simplicial complexes. There exists an integer $n \geq 0$ and a simplicial map $f : \mathbf{Sd}^n K \to L$ so that $|f|$ is homotopic to F.*

In general there is no known bound on how many barycentric subdivisions of K might be required to build the simplicial approximation f for a given F.

EXERCISES

EXERCISE 2.1. Prove that homotopy equivalence is an equivalence relation on the class of all topological spaces.

EXERCISE 2.2. Show that if two topological spaces X and Y are homeomorphic, then they must also be homotopy equivalent.

EXERCISE 2.3. Show that if Y is a contractible space, then then for any topological space X the product $X \times Y$ is homotopy equivalent to X.

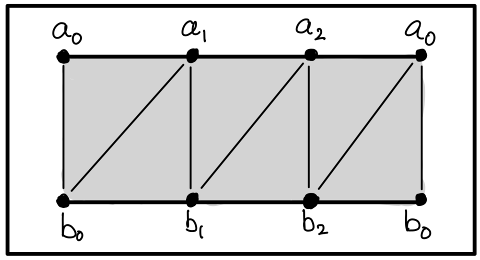
EXERCISE 2.4. Show that if Y is contractible then any pair of maps $f, g : X \to Y$ are homotopic.

EXERCISE 2.5. A subset $P \subset \mathbb{R}^n$ is said to be *convex* if for every pair of points x, y in P the line segment $\{tx + (1-t)y \mid 0 \leq t \leq 1\}$ lies inside P. Show that every nonempty convex set is contractible.

EXERCISE 2.6. Show that the subspaces $X = \{0\}$ and $Y = \{0, 1\}$ of the real line $\mathbb{R}$ are not homotopy equivalent.

EXERCISE 2.7. Prove the assertion **(a)** from Lemma 2.8. [Hint: the filtration $S_i K$, the inductive strategy and Lemma 2.5 are all useful here].

EXERCISE 2.8. Consider two simplicial maps from $\partial\Delta(2)$ to the illustrated simplicial complex described as follows. The first one sends vertices $\{0, 1, 2\}$ to $\{a_0, a_1, a_2\}$ in order, while the second one sends the same vertices to $\{b_0, b_1, b_2\}$ respectively. Show that these two maps are homotopic, keeping in mind that the left-most edge in the figure is identified with the right-most edge. [Hint: first show that the unit square $[0, 1] \times [0, 1]$ is contractible by Proposition 2.6 plus Exercise 2.3, and then apply Lemma 2.8]



EXERCISE 2.9. Given a simplicial map $f : K \to L$, show that for each simplex τ in L the fiber τ/f as defined in (2) is a subcomplex of K; also show that τ/f is a subcomplex of τ'/f whenever $\tau \leq \tau'$ holds in L.

EXERCISE 2.10. Find the smallest open cover of the circle with contractible supports. What is the nerve of this cover?

EXERCISE 2.11. Find a cover of the circle containing at least two open sets which violates the hypotheses of the nerve lemma. What is the nerve of this bad cover?

EXERCISE 2.12. Show that trees (connected graphs with no cycles) are simple homotopy-equivalent to $\Delta(0)$, and hence contractible. [Hint: induction on edges plus Proposition 2.14].

EXERCISE 2.13. Use a suitable sequence of elementary collapses to show that the simplicial complex drawn in Exercise 2.8 is homotopy-equivalent to the subcomplex consisting of the simplices $\{b_0, b_1, b_2, b_0b_1, b_1b_2, b_0b_2\}$.